COMPACTNESS OF THE NEUMANN-POINCARÉ OPERATOR

BY

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1. **Introduction.** The Neumann-Poincaré integral equation arises in connection with the Dirichlet and Neumann problems of potential theory and in connection with conformal mapping. Warschawski [6] has proved the compactness of the integral operator involved here under what seem to be natural smoothness conditions on the boundary curve, for the case where the boundary is a single contour. Because this proof relies heavily upon complex function theory, it does not extend easily to higher dimensions. It is the purpose of the present paper to give a proof, for the case of several contours, which will extend readily to higher dimensions.

2. **Definitions.** Let $\mathscr{B}_1, \ldots, \mathscr{B}_m$ be bounded nonintersecting contours in the plane whose interiors are disjoint, and let ζ_j be the standard representation[2] of $\mathscr{B}_j$. Let $s_0 = 0$, let s_j be the sum of the lengths of $\mathscr{B}_1, \ldots, \mathscr{B}_j$, and let $\mathscr{I} = [0, s_m]$. Let ζ be the function defined on $\mathscr{I}$ so that $\zeta s = \zeta_1 s$ for all s in $[0, s_1]$ and $\zeta s = \zeta_j(s - s_{j-1})$ for all s in $(s_{j-1}, s_j]$, $j = 2, \ldots, m$, and let $\mathscr{B}$ be the range of ζ.

Let A be the function defined for all ordered pairs (s, t) such that ζs and ζt both belong to $\mathscr{B}_j$ for some $j = 1, \ldots, m$ as follows:

$$\begin{aligned} A(s, t) &= s - t && \text{if } |s-t| < \tfrac{1}{2}(s_j - s_{j-1}); \\ &= (s-t) + (s_j - s_{j-1}) \operatorname{sgn}(t-s) && \text{if } |s-t| \geqq \tfrac{1}{2}(s_j - s_{j-1}). \end{aligned}$$

A function α defined on $\mathscr{I}$ will be said to satisfy a Hölder condition on $\mathscr{B}$ if and only if $|\alpha s - \alpha t| \leqq a|A(s, t)|^b$ for some numbers a and b such that $a > 0$ and $0 < b \leqq 1$. Continuity on $\mathscr{B}$ is defined analogously. Also, the derivative $D\alpha$ of α is that function whose value at t is $\lim_{A(s,t) \to 0} (\alpha s - \alpha t)/A(s, t)$.

It is assumed that ζ has a derivative $D\zeta$ which satisfies a Hölder condition on $\mathscr{B}$. If $\zeta = \xi + i\eta$, then $D\xi$ and $D\eta$ satisfy the same Hölder condition on $\mathscr{B}$ as does $D\zeta$.

3. **The space $\mathscr{P}$ and the operator T.** Let c be a number such that $c > |\zeta s - \zeta t|$ for all s and t in $\mathscr{I}$, and let the function Λ be defined on $\mathscr{I} \times \mathscr{I}$ (except at points (s, t) where $\zeta s = \zeta t$) by the equality $\Lambda(s, t) = \log (c/|\zeta s - \zeta t|)$. For any function α defined

Received by the editors July 8, 1968.

[1] This research was supported by the National Science Foundation under grant NSF G-4142 and by the Office of Naval Research under contract Nonr-3911(00).

[2] A contour $\mathscr{B}$ is the range of a mapping α of a closed interval $[a, b]$ into the plane such that α is continuous and of bounded variation, α is one-to-one on $[a, b)$ and $\alpha a = \alpha b$. If σt denotes the total variation of α on the interval $[a, t]$ for every t in $[a, b]$, and if σ^* is the inverse function of σ, then the composition $\alpha\sigma^*$ is the standard representation of $\mathscr{B}$.

on $\mathscr{I}$, let the function M be defined by the equality $M(s, t) = (\alpha s)(\alpha t)\Lambda(s, t)$. Then, if the double integral $\iint_{\mathscr{I}\times\mathscr{I}} M$ exists and is finite, it will be denoted by $\|\alpha\|^2$.

In subsequent proofs, use will be made of the fact that if $\|\alpha\|^2$ exists, then so do the corresponding iterated integrals, and the three are equal.

THEOREM 3.1. *If α is a continuous function on $\mathscr{I}$, then $\|\alpha\|^2 \geqq 0$. Moreover, $\|\alpha\|^2 = 0$ if and only if $\alpha = 0$.*

A proof of this theorem for the case of a single contour with continuous curvature is given on pages 157–159 of [1].

LEMMA 3.1. *If $\|\alpha\|^2$ exists and is finite, then α is summable on $\mathscr{I}$.*

Proof. If $\|\alpha\|^2$ exists, then $\int_0^{s_m} \alpha s\ \alpha t\ \Lambda(s, t)\, ds$ exists for almost all t in $\mathscr{I}$, which implies that, for almost all t in $\mathscr{I}$, the function $\alpha \cdot \Lambda(\iota, t)$, where ι is the identity function from $\mathscr{I}$ to $\mathscr{I}$, is summable on $\mathscr{I}$. Moreover, $1/\Lambda(\iota, t)$ is continuous on $\mathscr{I} - \{t\}$ and bounded there. Therefore α is the product of a summable function and a bounded measurable function; hence it is summable on $\mathscr{I}$.

THEOREM 3.2. *If $\|\alpha\|^2$ exists and is finite, then $\|\alpha\|^2 \geqq 0$.*

Proof. This theorem can be proved by making use of ideas in the proof of Lemma 1, p. 9, of [6].

It can be shown that, if $\|\alpha\|^2$ and $\|\beta\|^2$ exist and are finite, then the integral $\iint_{\mathscr{I}\times\mathscr{I}} P$, where $P(s, t) = (\alpha s)(\beta t)\Lambda(s, t)$, exists and is finite. This shows that the set of all functions α such that $\|\alpha\|^2$ exists and is finite can be regarded as an inner product space with the inner product $\langle \alpha, \beta \rangle$ given by the above integral. Let two functions α and β of this space be called equivalent if and only if $\|\alpha - \beta\| = 0$, and let a representative be chosen from each equivalence class. Let $\mathscr{P}$ be the inner product space of representatives.

To discuss the classical Neumann-Poincaré integral equation, it is convenient to introduce the linear operator $\boldsymbol{T}$ in $\mathscr{P}$ defined as follows. Let Γ be the function defined for each pair (x, y) of real numbers such that $x^2 + y^2 \neq 0$ by the equality

$$\Gamma(x, y) = \log (c/\sqrt{(x^2 + y^2)}),$$

let $D_{u_t}\Gamma$ be its directional derivative in the direction of the vector $u_t = -D\eta t + iD\xi t$, and let

$$K(s, t) = (1/\pi) D_{u_t}\Gamma(\xi s - \xi t, \eta s - \eta t),$$

for each ordered pair (s, t) in $\mathscr{I} \times \mathscr{I}$ except $(s_1, 0)$, $(0, s_1)$, and those for which $s = t$.

Since K is continuous on $\mathscr{I} \times \mathscr{I}$ except on a set of measure zero, it is measurable there. Moreover (see §6, Property (iii)), because the function whose value at s is $\int_{\mathscr{I}} |K(s, \iota)|$ satisfies a Hölder condition on $\mathscr{B}$, it is bounded and measurable on $\mathscr{I}$. If $\|\alpha\|^2$ exists, then, by Lemma 3.1, α is summable on $\mathscr{I}$, and it follows that the integral $\int_0^{s_m} \int_0^{s_m} |\alpha s K(s, t)|\, dt\, ds$ exists. Then, by Fubini's theorem, the function $\boldsymbol{T}\alpha$ whose value at t is $\int_{\mathscr{I}} (\alpha \cdot K(\iota, t))$, is summable on $\mathscr{I}$.

THEOREM 3.3. *If* $\|\alpha\|^2$ *exists, then* $\|T\alpha\|^2$ *exists.*

Proof. It is not difficult to show that the function $\bar{H}$ such that

$$\bar{H}(s, t) = \int_{\mathscr{I}} (|K(s, \iota)| \cdot \Lambda(\iota, t))$$

is continuous on $\mathscr{I} \times \mathscr{I}$ (see §6, Property (v)) and then that, for every function α for which $\|\alpha\|^2$ exists, the function β such that $\beta x = \int_{\mathscr{I}} (|\alpha| \cdot \bar{H}(\iota, x))$ is continuous on $\mathscr{I}$. If $\|\alpha\|^2$ exists, $T\alpha$ is summable on $\mathscr{I}$, and hence $\int_{\mathscr{I}} (\beta \cdot |T\alpha|)$ exists. But

$$\int_{\mathscr{I}} (\beta \cdot |T\alpha|) = \int_0^{s_m} \int_0^{s_m} \int_0^{s_m} |\alpha x| \, |K(x, t)| \, |T\alpha s| \Lambda(t, s) \, dt \, dx \, ds = \|T\alpha\|^2,$$

by virtue of a generalization of Fubini's theorem to three-place functions. This completes the proof.

Therefore T is an operator in $\mathscr{P}$. For $n = 2, 3, 4, \ldots$, $T^n\alpha t = \int_{\mathscr{I}} (\alpha \cdot K_n(\iota, t))$, where $K_n(s, t) = \int_{\mathscr{I}} (K_{n-1}(s, \iota) \cdot K(\iota, t))$ and $K_1 = K$. Since the function H for which $H(s, t) = \int_{\mathscr{I}} (K(s, \iota) \cdot \Lambda(\iota, t))$ has the property that $H(s, t) = H(t, s)$, as can be shown from Green's second identity, it follows that, for every α and β in $\mathscr{P}$, $\langle T\alpha, \beta\rangle = \langle \alpha, T\beta\rangle$.

4. **Definition and properties of Ω_n.** In classical potential theory (see, for example, [5, p. 299]), it is shown that there exists an orthonormal set $\{\varphi_1, \ldots, \varphi_m\}$ of functions such that $T\varphi_j = \varphi_j$ for $j = 1, \ldots, m$. Moreover, these functions have the properties that, for some nonzero real numbers $c_1, \ldots, c_m$,

$$(1) \qquad \int_{s_{j-1}}^{s_j} \varphi_k = c_k \quad \text{if } k = j; \qquad = 0 \quad \text{if } k \neq j,$$

for $k, j = 1, \ldots, m$, and

$$(2) \qquad \int_{\mathscr{I}} (\varphi_j \cdot \Lambda(s, \iota)) = 1/c_j \quad \text{if } \zeta s \in \mathscr{B}_j; \qquad = 0 \quad \text{if } \zeta s \in \mathscr{B}_k \text{ for } k \neq j,$$

for $j = 1, \ldots, m$. Since $\varphi_j = T^n\varphi_j$ for $j = 1, \ldots, m$ and for every positive integer n, it follows from Property (iv) (see §6) that the φ's satisfy a Hölder condition on $\mathscr{B}$.

THEOREM 4.1. *If n is a sufficiently large positive integer, there exists a function Ω_n such that, for each s and t in $\mathscr{I}$, $K_n(s, t) = \int_{\mathscr{I}} (\Omega_n(\iota, t) \cdot \Lambda(\iota, s))$, and, for each t in $\mathscr{I}$, $\Omega_n(\iota, t)$ is continuous on $\mathscr{B}$.*

For a proof of this theorem, see [6, p. 15].

Let I be the identity mapping of the complex plane onto itself, and, for each t in $\mathscr{I}$, let $\Phi_e'(I, t)$ be the solution of the exterior Dirichlet problem for $\mathscr{B}$ with boundary values $K_n(\iota, t)$. Since, for each t in $\mathscr{I}$, $D_1K_n(\iota, t)$ satisfies a Hölder condition on $\mathscr{B}$ if n is sufficiently large (see §6, Property (vii)), it follows (see [3, p. 111]) that for each (s, t) in $\mathscr{I} \times \mathscr{I}$, $\lim_{z \to \zeta s} D_{u_s} \Phi'_e(z, t)$ exists, and the function whose value at s is given by this limit is continuous on $\mathscr{B}$. Therefore, since the operator $\boldsymbol{T}$ is compact in $\mathscr{L}_2[0, s_m]$ (see [7, pp. 326–329]), it follows by the Fredholm theory that there is a function $\bar{\Omega}_n$ such that $\bar{\Omega}_n(\iota, t)$ is continuous on $\mathscr{B}$ and such that

$$\lim_{z \to \infty} \bar{\Phi}_e(z, t) = 0$$

and

$$\lim_{z \to \zeta s} D_{u_s} \bar{\Phi}_e(z, t) = \lim_{z \to \zeta s} \Phi'_e(z, t),$$

where

$$\bar{\Phi}_e(z, t) = \int_{\mathscr{I}} (\bar{\Omega}_n(\iota, t) \cdot \log (c/|z-\zeta|))$$

for all t in $\mathscr{I}$ and all z in $\mathscr{B}_e$, the unbounded region determined by $\mathscr{B}$. Since $\Phi'_e(I, t) - \bar{\Phi}_e(I, t)$ is a function which is harmonic in $\mathscr{B}_e$ and whose normal derivative is zero on $\mathscr{B}$, it follows by the uniqueness, to within an additive constant, of the solution of the Neumann problem, that there is a function γ on $\mathscr{I}$ to $\mathscr{I}$ such that

$$\Phi'_e(z, t) - \bar{\Phi}_e(z, t) = \gamma t$$

for all z in $\mathscr{B}_e$. An explicit expression for the function Ω_n whose existence is asserted in Theorem 4.1 is given by the equality

$$\Omega_n(\iota, t) = \bar{\Omega}_n(\iota, t) + (\gamma t)\omega,$$

where $\omega = \sum_{j=1}^{m} c_j \varphi_j$.

Let

$$\Phi_i(z, t) = \int_{\mathscr{I}} (\Omega_n(\iota, t) \cdot \log (c/|z-\zeta|))$$

for each z in $\mathscr{B}_i$, where $\mathscr{B}_i$ is the union of the interior regions determined by $\mathscr{B}$, let

$$\Psi_i(s, t) = \lim_{z \to \zeta s} D_{u_s} \Phi_i(z, t),$$

let

$$\Phi_e(z, t) = \int_{\mathscr{I}} (\Omega_n(\iota, t) \cdot \log (c/|z-\zeta|))$$

for each z in $\mathscr{B}_e$, and let

$$\Psi_e(s, t) = \lim_{z \to \zeta s} D_{u_s} \Phi_e(z, t).$$

Then, by the well-known behavior of the normal derivative of the potential due to a single-layer distribution,

$$\Omega_n = (1/2\pi)(\Psi_e - \Psi_i). \tag{3}$$

LEMMA 4.1. *There exist functions Δ_i and Δ_e such that, for all t in $\mathscr{I}$,*

$$\frac{1}{\pi}\int_0^{s_m} \Delta_i(s, t)D_{u_s}\Gamma(x-\xi s, y-\eta s)\, ds = \Phi_i(z, t) \tag{4}$$

for all z in $\mathscr{B}_i$, where $z=x+iy$; and

$$\begin{aligned}\frac{1}{\pi}\int_0^{s_m} &\Delta_e(s, t)D_{u_s}\Gamma(x-\xi s, y-\eta s)\, ds \\ &+\int_0^{s_m}\left(\sum_{j=1}^{m}(\varphi_j s)(\varphi_j t)-(\omega s)(\gamma t)\right)\log\frac{c}{|\zeta s - z|}\,ds+\gamma t = \Phi_e'(z, t)\end{aligned} \tag{5}$$

for all z in $\mathscr{B}_e$. Moreover, for each s in $\mathscr{I}$, $D_1\Delta_i(s, \iota)$, $D_1\Delta_e(s, \iota)$, $D_1\Delta_i(\iota, s)$, and $D_1\Delta_e(\iota, s)$ exist and satisfy Hölder conditions on $\mathscr{B}$, uniformly with respect to s.

Proof. Since $\lim_{z\to\zeta v}\Phi_i(z, t)=K_n(v, t)$, one can show, by using the well-known discontinuous behavior of a potential arising from a double-layer distribution and the uniqueness of the solution of the Dirichlet problem, that there exists a function Δ_i satisfying equation (4) if and only if the integral equation

$$\Delta_i(v, t)+\int_{\mathscr{I}}(\Delta_i(\iota, t)\cdot K(v, \iota)) = K_n(v, t) \tag{6}$$

has a solution. Since (see [7, pp. 326–329]) the operator $\boldsymbol{T}$ is compact in $\mathscr{L}_2[0, s_m]$, the Fredholm theory is applicable. The homogeneous equation corresponding to (6) has no nontrivial solutions, and hence for each t in $\mathscr{I}$, equation (6) has a unique solution $\Delta_i(\iota, t)$. It satisfies the equation

$$\Delta_i(v, t)+(-1)^{p+1}\int_{\mathscr{I}}(\Delta_i(\iota, t)\cdot K_p(v, \iota)) = \sum_{j=0}^{p-1}(-1)^j K_{n+j}(v, t) \tag{7}$$

for each positive integer p, as may be proved by induction on p, equation (6) being the statement for $p=1$. A solution of this equation for the case where $p=2n$ is given by the expression

$$\Delta_i(v, t) = \sum_{j=0}^{2n-1}(-1)^j\left[K_{n+j}(v, t)+\int_{\mathscr{I}}(K_{n+j}(\iota, t)\cdot P(v, \iota))\right],$$

where P is the Fredholm resolvent for the kernel K_{2n}. From this expression it follows that Δ_i is bounded on $\mathscr{I}\times\mathscr{I}$ and $\Delta_i(v, \iota)$ satisfies a Hölder condition on $\mathscr{B}$ uniformly in v, since the functions $\sum_{j=0}^{2n-1}(-1)^j K_{n+j}$ and P are both bounded on $\mathscr{I}\times\mathscr{I}$, and the function $\sum_{j=0}^{2n-1}(-1)^j K_{n+j}(v, \iota)$ satisfies a Hölder condition on $\mathscr{B}$ uniformly in v (see §6, Property (iv)). Moreover, if p is taken to be $2n$ in equation (7), it follows that, for each v in $\mathscr{I}$, $D_1\Delta_i(v, \iota)$ exists and

$$D_1\Delta_i(v, t) = \int_{\mathscr{I}}(\Delta_i(\iota, t)\cdot D_1K_{2n}(v, \iota))+\sum_{j=0}^{2n-1}(-1)^j D_1K_{n+j}(v, t), \tag{8}$$

differentiation under the integral sign being justified because $\Delta^i(\iota, t)\cdot D_1K_{2n}(v, \iota)$ is summable for all v and t in $\mathscr{I}$. Now from the facts that D_1K_{2n} is bounded on $\mathscr{I}\times\mathscr{I}$ and that $D_1K_n(v, \iota), \dots, D_1K_{3n-1}(v, \iota)$, and $\Delta_i(v, \iota)$ satisfy Hölder conditions on $\mathscr{B}$ uniformly with respect to v, it follows by (8) that $D_1\Delta_i(v, \iota)$ satisfies a Hölder condition on $\mathscr{B}$, uniformly with respect to v.

Furthermore, since Δ_i is bounded on $\mathscr{I}\times\mathscr{I}$ and $D_1K_n(\iota, t), \dots, D_1K_{3n-1}(\iota, t)$ satisfy Hölder conditions on $\mathscr{B}$ uniformly with respect to t (see §6, Property (iii)), it follows from (8) that $D_1\Delta_i(\iota, t)$ satisfies a Hölder condition on $\mathscr{B}$, uniformly with respect to t. Similarly, $D_1\Delta_i$ is bounded on $\mathscr{I}\times\mathscr{I}$. This completes the proof of the statements about Δ_i in the lemma.

A similar treatment of Φ'_e is impossible because the integral equation corresponding to (6) in this case would be

$$\Delta_e(v, t)-\int_{\mathscr{I}}(\Delta_e(\iota, t)\cdot K(v, \iota)) = -K_n(v, t),$$

which has no solution because the associated homogeneous equation

$$\Delta_e(v, t)-\int_{\mathscr{I}}(\Delta_e(\iota, t)\cdot K(\iota, v)) = 0$$

has solutions which are not orthogonal in $\mathscr{L}_2[0, s_m]$ to the function $-K_n(\iota, t)$. However, the integral equation

$$\Delta_e(v, t)-\int_{\mathscr{I}}(\Delta_e(\iota, t)\cdot K(v, \iota)) = -K_n(v, t)+\sum_{j=1}^{m}(\mu_j v)(\varphi_j t),$$

where $\mu_j v=\int_{\mathscr{I}}(\varphi_j\cdot\Lambda(v, \iota))$, does have a solution $\Delta_e(\iota, t)$, which then has the properties that

$$\lim_{z\to\zeta v}\Xi(z, t) = K_n(v, t)-\sum_{j=1}^{m}(\mu_j v)(\varphi_j t)$$

and $\lim_{z\to\infty}\Xi(z, t)=0$, where

$$\Xi(z, t) = (1/\pi)\int_0^{s_m}\Delta_e(s, t)D_{u_s}\Gamma(x-\xi s, y-\eta s)\,ds,$$

$z=x+iy$. Hence the function whose value at z is

$$\Xi(z, t)+\sum_{j=1}^{m}(\varphi_j t)\int_{\mathscr{I}}(\varphi_j\cdot\log(c/|\zeta-z|)),$$

has the same limit on $\mathscr{B}$ as does $\Phi'_e(I, t)$, but its limit at infinity is zero, whereas $\lim_{z\to\infty}\Phi'_e(z, t)=\gamma t$. From these considerations, equation (5) follows.

As can be proved by induction, $\Delta_e(\iota, t)$ also satisfies the equation

$$\Delta_e(v, t)-\int_{\mathscr{I}}(\Delta_e(\iota, t)\cdot K_p(v, \iota)) = -\sum_{k=0}^{p-1}K_{n+k}(v, t)+p\sum_{j=1}^{m}(\mu_j v)(\varphi_j t),$$

for each positive integer p. A solution of this equation for the case where $p=2n$ is given by the relation

$$\Delta_e(v,t) = -\sum_{k=0}^{2n-1} K_{n+k}(v,t)+2n\sum_{j=1}^{m}(\mu_j v)(\varphi_j t) + \int_{\mathscr{I}}\left(\left(\sum_{k=0}^{2n-1} K_{n+k}(\iota,t)+2n\sum_{j=1}^{m}(\varphi_j t)\mu_j\right)\cdot P(v,\iota)\right).$$

The argument to show that $D_1\Delta_e$ has the properties stated in the lemma now proceeds like that for $D_1\Delta_i$.

LEMMA 4.2. *For each s in $\mathscr{I}$, $\Omega_n(s,\iota)$ is continuous on $\mathscr{B}$, and the continuity is uniform with respect to s.*

Proof. For each (s,t) in $\mathscr{I}\times\mathscr{I}$,

$$\Psi_i(s,t) = \int_{\mathscr{I}}(D_1\Delta_i(\iota,t)\cdot E(\iota,s))$$

and

$$\Psi_e(s,t) = \int_{\mathscr{I}}(D_1\Delta_e(\iota,t)\cdot E(\iota,s))+\sum_{j=1}^{m}(\varphi_j s)(\varphi_j t),$$

where

$$E(v,s) = (1/2\pi)[(\xi s-\xi v)(D\xi s)+(\eta s-\eta v)(D\eta s)]/[(\xi s-\xi v)^2+(\eta s-\eta v)^2],$$

and where the Cauchy principal value of each integral is understood (see [2, p. 46]). Now if r is a sufficiently small positive number, there exist functions θ_1 and θ_2, defined on the set

$$\{v : 0 < |A(v,s)| \leqq r\},$$

such that

$$|\theta_1 v-s| < |A(v,s)|, \qquad |\theta_2 v-s| < |A(v,s)|,$$

and

$$E(v,s) = Z(v,s)/A(s,v),$$

where

$$Z(v,s) = [(D\xi\theta_1 v)(D\xi s)+(D\eta\theta_2 v)(D\eta s)]/[(D\xi\theta_1 v)^2+(D\eta\theta_2 v)^2].$$

Also, for each sufficiently small positive number r, there is a positive number κr such that $|E(w,s)|<\kappa r$ for all (w,s) in the set

$$\mathscr{I}\times\mathscr{I}-\{(w,s) : |A(w,s)| \leqq r\}.$$

For each s in $\mathscr{I}$, let β_s be the inverse of the function $A(s,\iota)$, and let $v_1=\beta_s(-r)$ and $v_2=\beta_s r$. Then, by Lemma 4.1, there exist numbers f_1 and g_1 such that $f_1>0$, $0<g_1\leqq 1$, and for all h for which $(t+h)\in\mathscr{I}$,

$$|\Psi_i(s,t+h)-\Psi_i(s,t)| \leqq 2(\kappa r)f_1 s_m|A(t+h,t)|^{g_1} + \left|(cpv)\int_{v_1}^{v_2}[(D_1\Delta_i(\iota,t+h)-D_1\Delta_i(\iota,t))\cdot E(\iota,s)]\right|.$$

Since $D\xi$, $D\eta$, and $D_1\Delta_i(\iota, t)$ satisfy Hölder conditions on $\mathscr{B}$ and since $(D\xi)^2 + (D\eta)^2 = 1$, it can be shown by standard arguments (see, for example, [4, §6, 2° and 3°]) that, for r sufficiently small, for some positive number d_1 and some d_2 such that $0 < d_2 \leqq 1$, for all s in $\mathscr{S}$, and for all u such that $0 < u \leqq r$,

$$|D_1\Delta_i(\beta_s(-u), t)Z(\beta_s(-u), s) - D_1\Delta_i(\beta_s u, t)Z(\beta_s u, s)| \leqq d_1|2u|^{d_2},$$

whence

$$\begin{aligned} &\left| (cpv) \int_{v_1}^{v_2} (D_1\Delta_i(\iota, t)\cdot E(\iota, s)) \right| \\ &\quad = \lim_{q\to 0+} \left| \int_{v_1}^{\beta_s(-q)} (D_1\Delta_i(\iota, t)\cdot E(\iota, s)) + \int_{\beta_s q}^{v_2} (D_1\Delta_i(\iota, t)\cdot E(\iota, s)) \right| \\ &\quad = \lim_{q\to 0+} \left| \int_q^r [D_1\Delta_i(\beta_s(-u), t)Z(\beta_s(-u), s) - D_1\Delta_i(\beta_s u, t)Z(\beta_s u, s)] \frac{1}{u}\, du \right| \\ &\quad \leqq \lim_{q\to 0+} \frac{2^{d_2}\, d_1(r^{d_2} - q^{d_2})}{d_2} = d_1(2r)^{d_2}/d_2. \end{aligned}$$

The integral

$$\left| (cpv) \int_{v_1}^{v_2} (D_1\Delta_i(\iota, t+h)\cdot E(\iota, s)) \right|,$$

can be treated in exactly the same way. Therefore

$$|\Psi_i(s, t+h) - \Psi_i(s, t)| \leqq 2(\kappa r)f_1 s_m |h|^{g_1} + 2d_1(2r)^{d_2}/d_2.$$

By choosing r sufficiently small, and then h, it is possible to make

$$|\Psi_i(s, t+h) - \Psi_i(s, t)|$$

arbitrarily small, and the choice of h is independent of s. This proves that $\Psi_i(s, \iota)$ is continuous and that the continuity is uniform with respect to s. The proof that $\Psi_e(s, \iota)$ has the same property is similar. The lemma then follows from equation (3).

LEMMA 4.3. *The function whose value at t is $\int_{\mathscr{S}} |\Omega_n(\iota, t)|$ is continuous and bounded on $\mathscr{S}$.*

Proof. This follows from the uniform continuity in Lemma 4.2.

LEMMA 4.4. *For some positive number a_0, $\|\Omega_n(\iota, t)\| \leqq a_0$ for all t in $\mathscr{S}$. For some numbers f and g such that $f > 0$ and $0 < g \leqq 1$, for all t and $t+h$ in $\mathscr{S}$,*

$$\|\Omega_n(\iota, t+h) - \Omega_n(\iota, t)\| \leqq f|A(t+h, t)|^g.$$

Proof. The first inequality follows from Lemma 4.3 and the inequality

$$\begin{aligned} \|\Omega_n(\iota, t)\|^2 &= \int_0^{s_m}\int_0^{s_m} \Omega_n(u, t)\Omega_n(v, t)\Lambda(u, v)\, du\, dv \\ &= \int_0^{s_m} \Omega_n(u, t)K_n(u, t)\, du \leqq k_n \int_0^{s_m} |\Omega_n(u, t)|\, du, \end{aligned}$$

where k_n is a number such that $|K_n(u, t)| \leqq k_n$ (see §6, Property (i)).

The second part of the lemma follows from Lemma 4.2 and the fact that

$$\begin{aligned}\|\Omega_n(\iota, t+h)-\Omega_n(\iota, t)\|^2 &= \int_{\mathscr{S}} [\Omega_n(\iota, t+h)-\Omega_n(\iota, t))\cdot(K_n(\iota, t+h)-K_n(\iota, t))] \\ &\leqq b_0|A(t+h, t)|^g \int_{\mathscr{S}} |\Omega_n(\iota, t+h)-\Omega_n(\iota, t)|,\end{aligned} \tag{9}$$

where the fact that $K_n(s, \iota)$ satisfies a Hölder condition on $\mathscr{B}$ has been exploited.

5. **Properties of T.**

LEMMA 5.1. *The operator T is bounded on $\mathscr{P}$.*

Proof. For n sufficiently large, for each α in $\mathscr{P}$ and each t in $\mathscr{S}$,

$$|T^n\alpha t| = |\langle\Omega_n(\iota, t), \alpha\rangle| \leqq \|\Omega_n(\iota, t)\|\cdot\|\alpha\| \leqq a_0\|\alpha\| \tag{10}$$

by Lemma 4.4. Therefore

$$\|T^n\alpha\|^2 \leqq \int_0^{s_m}\int_0^{s_m} |T^n\alpha s|\,|T^n\alpha t|\Lambda(s, t)\,ds\,dt \leqq a_0^2\|\alpha\|^2\int_0^{s_m}\int_0^{s_m}\Lambda(s, t)\,ds\,dt,$$

which shows that T^n is bounded in $\mathscr{P}$ for all sufficiently large n. In particular, T^{2^p} is bounded for sufficiently large p, from which it follows that T is bounded on $\mathscr{P}$ by virtue of the fact that

$$\|T^{2^p-1}\alpha\|^2 = \langle T^{2^p-1}\alpha, T^{2^p-1}\alpha\rangle = \langle T^{2^p}\alpha, \alpha\rangle \leqq \|T^{2^p}\alpha\|\cdot\|\alpha\|$$

for all α in $\mathscr{P}$.

Now let $\mathscr{H}$ be the Hilbert space obtained by completing the inner product space $\mathscr{P}$, and let the extension of T to $\mathscr{H}$ by continuity be denoted by the same symbol.

THEOREM 5.1. *The operator T is compact in $\mathscr{H}$.*

Proof. Let $\mathscr{F}$ be any bounded set in $\mathscr{H}$, so that, for some positive number c_0, $\|\alpha\| \leqq c_0$ for all α in $\mathscr{F}$. Let $\mathscr{G} = T^{2n}\mathscr{F}$, where n is the integer introduced in Theorem 4.1. Then every element of $\mathscr{G}$ is a function continuous on $\mathscr{B}$. To see this, let α be any element of $\mathscr{F}$. Then there exists a sequence of functions $\beta_1, \beta_2, \beta_3, \ldots$ in $\mathscr{P}$ such that $\lim_{k\to\infty}\beta_k = \alpha$. Moreover, $T^n\alpha = \lim_{k\to\infty} T^n\beta_k$. Because this sequence converges, there exists a $\bar{c}_0$ such that $\|\beta_k\| \leqq \bar{c}_0$ for $k = 1, 2, 3, \ldots$, and hence, for all t in $\mathscr{S}$, $|T^n\beta_k t| \leqq a_0\bar{c}_0$, by virtue of the relation (10). Then, for all s and $s+h$ in $\mathscr{S}$,

$$\begin{aligned}|T^{2n}\beta_k(s+h) - T^{2n}\beta_k s| &= \left|\int_{\mathscr{S}} [(K_n(\iota, s+h)-K_n(\iota, s))\cdot(T^n\beta_k)]\right| \\ &\leqq a_0 b_0 \bar{c}_0 |A(s+h, s)|^g,\end{aligned}$$

where the same Hölder condition as was exploited in (9) has been used here. From the fact that

$$|T^{2n}\beta_k t - T^{2n}\beta_j t| \leqq \|\Omega_{2n}(\iota, t)\|\cdot\|\beta_k - \beta_j\| \leqq a_0\|\beta_k - \beta_j\|,$$

it follows that the sequence of continuous functions $T^{2n}\beta_1$, $T^{2n}\beta_2$, $T^{2n}\beta_3, \ldots$ is uniformly convergent, so that the limit function is continuous on $\mathscr{B}$. Since this pointwise convergence implies convergence in the norm, the limit function is $T^{2n}\alpha$.

Now the functions of $\mathscr{G}$ are uniformly bounded because, by the relation (10), $|T^{2n}\alpha t| \leqq a_0' c_0$, for some positive number a_0'. Moreover, $\mathscr{G}$ is equicontinuous because, for each α in $\mathscr{F}$, and each t and $t+h$ in $\mathscr{I}$,

$$|T^{2n}\alpha(t+h) - T^{2n}\alpha t| \leqq \|\Omega_{2n}(\iota, t+h) - \Omega_{2n}(\iota, t)\| \cdot \|\alpha\| \leqq c_0 f |A(t+h, t)|^g$$

by Lemma 4.4. By Ascoli's theorem, every sequence of functions in $\mathscr{G}$ contains a pointwise convergent subsequence, which subsequence also converges in the norm. This shows that T^{2n} is compact in $\mathscr{H}$. Since T is self-adjoint, it follows (see, for example, [7, p. 317], that T is compact in $\mathscr{H}$.

From the first part of this proof, it follows that every characteristic vector of T is a function which satisfies a Hölder condition on $\mathscr{B}$ because every characteristic vector of T is also a characteristic vector of T^{2n}.

6. **Properties of** K. For convenience, certain properties of K and its iterates are listed here. The set $\mathscr{S}$ is the set obtained by removing the points $(0, s_1)$, $(s_1, 0)$, and all points of the diagonal (i.e., points of the form (s, s)) from the set $\mathscr{I} \times \mathscr{I}$. The number b is the exponent of the Hölder condition satisfied by $D\zeta$.

(i) For each positive integer n, if $1 - nb > 0$, then $K_n \cdot |A|^{1-nb}$ is bounded on $\mathscr{S}$; if $1 - nb < 0$, then K_n is bounded on $\mathscr{S}$. Hence, for sufficiently large n, it is possible to define K_n at all points of $\mathscr{I} \times \mathscr{I}$ by continuity.

(ii) For each positive integer n, $D_1 K_n$ exists and, if
$1-(n-1)b > 0$, then $(D_1 K_n) \cdot |A|^{2-nb}$ is bounded on $\mathscr{S}$; if
$1-(n-1)b < 0$, then $(D_1 K_n) \cdot |A|^{2-b}$ is bounded on $\mathscr{S}$.

(iii) If α is any function which is bounded and measurable on $\mathscr{I}$, then the function whose value at t is $\int_{\mathscr{I}} (K_n(t, \iota) \cdot \alpha)$ satisfies a Hölder condition on $\mathscr{B}$ with exponent $\min\{1, nb\}$, for each positive integer n. In particular, if q is a positive integer such that K_q is bounded on $\mathscr{S}$, then $K_{q+n}(\iota, s)$ satisfies a Hölder condition on $\mathscr{B}$ with exponent $\min\{1, nb\}$ for all s in $\mathscr{I}$.

(iv) If α is any function which is bounded and measurable on $\mathscr{I}$, then the function whose value at s is $\int_{\mathscr{I}} (K(\iota, s) \cdot \alpha)$ satisfies a Hölder condition on $\mathscr{B}$ with exponent b', where b' is any number such that $0 < b' < b$. In particular, if q is a positive integer such that K_q is bounded on $\mathscr{S}$, then $K_{q+n}(s, \iota)$ satisfies a Hölder condition on $\mathscr{B}$ with exponent b' for every positive integer n and for every s in $\mathscr{I}$.

(v) There exist a_1, a_2, b_1, and b_2 such that $a_1 > 0$, $a_2 > 0$, $0 < b_1 < 1$, $0 < b_2 < 1$, and

$$|H(s+h, t+k) - H(s, t)| \leqq a_1 |A(s+h, s)|^{b_1} + a_2 |A(t+k, t)|^{b_2}$$

for all (s, t) in $\mathscr{I} \times \mathscr{I}$ and all h and k sufficiently close to zero. This statement is also true if H is replaced by $\overline{H}$.

(vi) For each positive integer n, $D_1K_n(s, t) = (-1)^{n+1}D_1K_n(t, s)$ for all points (s, t) of $\mathscr{S}$.

(vii) If n is a sufficiently large positive integer, then $D_1K_n(\iota, s)$ and $D_1K_n(s, \iota)$ each satisfy a Hölder condition on $\mathscr{B}$ with exponent b', where b' is any number such that $0 < b' < b$, for all s in $\mathscr{I}$.

Properties (i) and (ii) can be proved by induction. The proofs for the case $n = 1$ follow from the equality

$$K(s, t) = \frac{1}{\pi}\frac{(\eta s - \eta t)\, D\xi t - (\xi s - \xi t)\, D\eta t}{(\xi s - \xi t)^2 + (\eta s - \eta t)^2}. \tag{11}$$

The induction argument for Property (ii) outlined by Warschawski (see [6, p. 12]) for the case $m = 1$ extends easily to the case of several contours. Properties (i) and (ii) can then be used to prove (iii).

Property (iv) can be proved as follows. Suppose that $|\alpha s| < c_0$ for all s in $\mathscr{I}$ and that ζt and $\zeta(t+k)$ both belong to $\mathscr{B}_j$, and let r be a positive number such that $r < \frac{1}{2}(s_j - s_{j-1})$. Let k be any number such that $2|k| < r$, and let

$$\mathscr{E}_1 = \{s : 2|k| \leqq |A(s, t)| \leqq r\}, \qquad \mathscr{E}_2 = \{s : |A(s, t)| \leqq 2|k|\}.$$

Then

$$\left| \int_{\mathscr{I}} (K(\iota, t+k)\cdot\alpha) - \int_{\mathscr{I}} (K(\iota, t)\cdot\alpha) \right|$$
$$\leqq c_0\left[\int_{\mathscr{I}-(\mathscr{E}_1\cup\mathscr{E}_2)} \Theta(\iota, t, k) + \int_{\mathscr{E}_1} \Theta(\iota, t, k) + \int_{\mathscr{E}_2} \Theta(\iota, t, k)\right],$$

where $\Theta(s, t, k) = |K(s, t+k) - K(s, t)|$. From the expression (11) for K, it follows by standard arguments that the first integral on the right is equal to or less than the product of $|A(t+k, t)|^b$ and some positive number.

For all (s, t) such that $|A(s, t)| \leqq r$, let

$$X(s, t) = \frac{\xi s - \xi t}{A(s, t)} \quad \text{if } A(s, t) \neq 0, \qquad Y(s, t) = \frac{\eta s - \eta t}{A(s, t)} \quad \text{if } A(s, t) \neq 0,$$
$$= D\xi s \quad \text{if } A(s, t) = 0, \qquad\qquad = D\eta s \quad \text{if } A(s, t) = 0.$$

Then, for each s in $\mathscr{E}_1 \cup \mathscr{E}_2$, $X(s, \iota)$, $X(\iota, s)$, $Y(s, \iota)$, and $Y(\iota, s)$ satisfy a Hölder condition on $\mathscr{E}_1 \cup \mathscr{E}_2$ with exponent b (see [4, p. 20]). Furthermore, $K = N/\pi A$, where

$$N(s, t) = [Y(s, t)D\xi t - X(s, t)D\eta t]/[X^2(s, t) + Y^2(s, t)].$$

Since, for some positive number d_0 and for all (s, t) in $\mathscr{I} \times \mathscr{I}$, $X^2(s, t) + Y^2(s, t) > d_0$, it follows that $N(s, \iota)$ and $N(\iota, s)$ satisfy a Hölder condition on $\mathscr{E}_1 \cup \mathscr{E}_2$ with exponent b, for each s in $\mathscr{E}_1 \cup \mathscr{E}_2$.

Since $A(s, \iota)$ satisfies a Lipschitz condition for each s in $\mathscr{E}_1$, it follows by Property (i) and Schwarz's inequality that

$$\int_{\mathscr{E}_1} \Theta(\iota, t, k) = \frac{1}{\pi} \int_{\mathscr{E}_1} \left| \frac{N(\iota, t+k)}{A(\iota, t+k)} - \frac{N(\iota, t)}{A(\iota, t)} \right| \leqq c_1 |A(t+k, t)|^b \log |A(t+k, t)|,$$

for some positive number c_1.

By Property (i), for some positive numbers c_2 and c_3,

$$\int_{\mathscr{E}_2} \Theta(\iota, t, k) \leqq \int_{\mathscr{E}_2} \left(\frac{c_2}{|A(\iota, t+k)|^{1-b}} + \frac{c_2}{|A(\iota, t)|^{1-b}} \right) \leqq c_3 |A(t+k, t)|^b,$$

which completes the proof of Property (iv).

Property (v) can be proved as follows.

$$\tag{12} \begin{aligned} |H(s+h, t+k) - H(s, t)| &\leqq \int_{\mathscr{I}} (|K(s+h, \iota) + K(s, \iota)| \cdot |\Lambda(\iota, t+k)|) \\ &\quad + \int_{\mathscr{I}} (|K(s, \iota)| \cdot |\Lambda(\iota, t+k) - \Lambda(\iota, t)|) \\ &\leqq \left(\int_{\mathscr{I}} |K(s+h, \iota) - K(s, \iota)|^{p_1} \right)^{1/p_1} \left(\int_{\mathscr{I}} |\Lambda(\iota, t+k)|^{q_1} \right)^{1/q_1} \\ &\quad + \left(\int_{\mathscr{I}} |K(s, \iota)|^{p_2} \right)^{1/p_2} \left(\int_{\mathscr{I}} |\Lambda(\iota, t+k) - \Lambda(\iota, t)|^{q_2} \right)^{1/q_2} \end{aligned}$$

for properly chosen numbers p_1, q_1, p_2, q_2 such that $1/p_1 + 1/q_1 = 1$, $1/p_2 + 1/q_2 = 1$, $1 \leqq p_1$, and $1 \leqq p_2$.

The integral $\int_{\mathscr{I}} |K(s+h, \iota) - K(s, \iota)|^{p_1}$ may now be treated like the integral $\int_{\mathscr{I}} \Theta(\iota, t, k)$ in the proof of (iv) above. The result is that, for some positive number d_1,

$$\left(\int_{\mathscr{I}} |K(s+h, \iota) - K(s, \iota)|^{p_1} \right)^{1/p_1} \leqq d_1 |A(s+h, s)|^{(1/p_1) - (1-b)},$$

provided that $p_1(1-b) < 1$.

Now $\Lambda(s, t) = B(s, t) + \log (1/|A(s, t)|)$, where

$$B(s, t) = \log (c/\sqrt{(X^2(s, t) + Y^2(s, t))}).$$

Since B is continuous on $\mathscr{I} \times \mathscr{I}$ and $|\log (1/|A(\iota, t)|)|^{q_1}$ is summable on $\mathscr{I}$ for every positive number q_1, it follows by the triangle inequality in $\mathscr{L}_{q_1}$ that

$$\left(\int_{\mathscr{I}} |\Lambda(\iota, t+k)|^{q_1} \right)^{1/q_1}$$

exists and is finite.

Property (i) can be used to show that $\left(\int_{\mathscr{I}} |K(s, \iota)|^{p_2}\right)^{1/p_2}$ exists and is finite for every p_2 such that $p_2(1-b) < 1$.

The last integral in inequality (12) can be treated as follows:

$$\left(\int_{\mathscr{I}} |\Lambda(\iota, t+k)-\Lambda(\iota, t)|^{q_2}\right)^{1/q_2} \leqq \left(\int_{\mathscr{I}} |B(\iota, t+k)-B(\iota, t)|^{q_2}\right)^{1/q_2} + \left(\int_{\mathscr{I}} \left|\log\left|\frac{A(\iota, t)}{A(\iota, t+k)}\right|\right|^{q_2}\right)^{1/q_2}.$$

Since $X(s, \iota)$ and $Y(s, \iota)$ satisfy a Hölder condition with exponent b, so does $B(s, \iota)$, and hence

$$\left(\int_{\mathscr{I}} |B(\iota, t+k)-B(\iota, t)|^{q_2}\right)^{1/q_2} \leqq d_2|A(t+k, t)|^b$$

for some positive number d_2. Now the second integral above may be written as a sum of integrals over $\mathscr{E}_1$, $\mathscr{E}_2$, and $\mathscr{I}-(\mathscr{E}_1 \cup \mathscr{E}_2)$. Exploiting the Lipschitz condition satisfied by $A(s, \iota)$ in the first of these three integrals, applying Minkowski's inequality to the second, and making use of a well-known inequality for the logarithm function in the third gives that

$$\left(\int_{\mathscr{I}} \left|\log\left|\frac{A(\iota, t)}{A(\iota, t+k)}\right|\right|^{q_2}\right)^{1/q_2} \leqq d_3(|A(t+k, t)|^{1-b_3})^{1/q_2}$$

for some positive number d_3 and some b_3 such that $0<b_3<1$.

The proof that $\bar{H}$ satisfies a similar condition is almost exactly like that for H. This completes the proof of Property (v).

Property (vi) can be proved by induction. That it is true when $n=1$ can be seen from the expression obtained by differentiating the expression (11) for K. Since

$$K_n(s, t) = \int_{\mathscr{I}} (K(s, \iota)\cdot K_{n-1}(\iota, t)) = \int_{\mathscr{I}} (K_{n-1}(s, \iota)\cdot K(\iota, t))$$

and

$$\int_{\mathscr{I}} K_p(s, \iota) = 1,$$

for each positive integer p, it follows that

$$D_1K_n(s, t) = \int_{\mathscr{I}} [D_1K(s, \iota)\cdot(K_{n-1}(\iota, t)-K_{n-1}(s, t))]$$
$$= \int_{\mathscr{I}} [D_1K_{n-1}(s, \iota)\cdot(K(\iota, t)-K(s, t))].$$

From the induction assumption and the symmetry of D_1K, it then follows by an integration by parts that

$$D_1K_n(t, s) = \int_{\mathscr{I}} [D_1K(\iota, t)\cdot(K_{n-1}(\iota, s)-K_{n-1}(t, s))]$$
$$= (-1)^{n+1}\int_{\mathscr{I}} [D_1K_{n-1}(s, \iota)\cdot(K(\iota, t)-K(s, t))] = (-1)^{n+1}D_1K_n(s, t).$$

The integration by parts is justified because the function

$$(K(\iota, t)-K(s, t))\cdot(K_{n-1}(\iota, s)-K_{n-1}(t, s))$$

is continuous on $\mathscr{B}$, and its derivative exists and is continuous except possibly at s and t, and its derivative is summable on $\mathscr{I}$.

Property (vii) can now be proved as follows. By Properties (i) and (ii), for p and q sufficiently large, $D_1K_p(s, \iota)$ is summable and $K_q(\iota, t)$ is bounded, so that

$$D_1K_{p+q}(s, t) = \int_{\mathscr{I}} (D_1K_p(s, \iota)\cdot K_q(\iota, t)),$$

and hence D_1K_{p+q} is bounded. Therefore, by Property (iv), $D_1K_{p+q+1}(s, \iota)$ satisfies a Hölder condition on $\mathscr{B}$ because

$$D_1K_{p+q+1}(s, t) = \int_{\mathscr{I}} (D_1K_{p+q}(s, \iota)\cdot K(\iota, t)).$$

The fact that $D_1K_{p+q+1}(\iota, t)$ also satisfies a Hölder condition on $\mathscr{B}$ then follows from Property (vi).

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